A REMARK ON THE ADDITIVITY OF TRACES IN TRIANGULATED CATEGORIES

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§1. Introduction and statements

In what follows a tensor category is understood to be an ACU $\otimes$-category in the sense of Saavedra Rivano [5, Ch. I, 2.4.1]. We denote the unit object by $\mathbb{1}$, the commutativity constraint by ψ, and the tensor structure by $\otimes$. There is also an associativity constraint that we omit and all these constraints are subject to natural compatibility conditions (*loc. cit.* I, 2.4). Recall (Deligne [2, 2.1.2]) that an object X of a tensor category is said to be dualizable if there is an object $X^\vee$ and morphisms $\delta_X\colon \mathbb{1} \to X \otimes X^\vee$ and $\mathrm{ev}_X\colon X^\vee \otimes X \to \mathbb{1}$ such that the diagrams

$$
\begin{tikzcd}
\mathbb{1} \otimes X \arrow[rr, "\delta_X \otimes \mathrm{id}_X"] \arrow[drr, "\psi_{\mathbb{1},X}"'] & & X \otimes X^\vee \otimes X \arrow[d, "\mathrm{id}_X \otimes \mathrm{ev}_X"] \\
& & X \otimes \mathbb{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X^\vee \otimes X \otimes X^\vee \arrow[d, "\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}"'] & & X^\vee \otimes \mathbb{1} \arrow[ll, "\mathrm{id}_{X^\vee} \otimes \delta_X"'] \arrow[dll, "\psi_{X^\vee,\mathbb{1}}"] \\
\mathbb{1} \otimes X^\vee & &
\end{tikzcd}
$$

are commutative. For example for the tensor category of modules over a commutative ring, dualizability is (e.g. *loc. cit.* 2.6) the same as being finitely generated and projective. With an appropriate interpretation, the morphism ev_X gives the trace. More concretely, let X be a dualizable object and $f\colon X \to X$ an endomorphism. The trace of f, here denoted by $\mathrm{tr}(f;X)$, is defined to be the composite

$$\mathbb{1} \xrightarrow{\delta_X} X \otimes X^\vee \xrightarrow{f\otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \xrightarrow{\psi_{X,X^\vee}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}.$$

This is an element of $\mathrm{End}(\mathbb{1})$. The resulting map $\mathrm{tr}\colon \mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$ is linear. Moreover, when defined, the trace $\mathrm{tr}(f \otimes g; X \otimes Y)$ is the product of $\mathrm{tr}(f;X)$ and $\mathrm{tr}(g;Y)$. For the proofs of these and other properties see any of the references cited above.

We clarify some terminologies. A tensor category as above is (Mac Lane [3]) also called an (additive) symmetric monoidal category. A symmetric monoidal category in which each functor $Z \mapsto Z \otimes X$ has a right adjoint is (Eilenberg-Kelly [1]) said to be closed. Recall the following result.

Theorem 1.1 (May [4, 0.1])**.–** *For any distinguished triangle $\Delta : X \to Z \to Y \to X[1]$ of dualizable objects in a closed symmetric monoidal category with a compatible triangulation we have*

$$\mathrm{tr}(\mathrm{id};Z) = \mathrm{tr}(\mathrm{id};X) + \mathrm{tr}(\mathrm{id};Y).$$

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In what follows we let D be a k-linear Karoubian (i.e. pseudo-abelian) rigid tensor triangulated category where $k = \bar{k}$ is an algebraically closed field of characteristic zero. Note that linearity means ([5, Ch. I, 0.1.2]) that $\mathrm{End}(\mathbb{1})$ is a k-algebra. Here the term *rigid tensor triangulated* means a closed symmetric monoidal category with a compatible triangulation in the sense of [4] and in which every object is dualizable.

An endomorphism $f = (f_X, f_Z, f_Y)$ of a distinguished triangle Δ in D is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f_X"] & Z \arrow[r] \arrow[d, "f_Z"] & Y \arrow[r] \arrow[d, "f_Y"] & X[1] \arrow[d, "f_X[1]"] \\ X \arrow[r] & Z \arrow[r] & Y \arrow[r] & X[1] \end{tikzcd} \tag{1}$$

with both rows being the given triangle Δ. For example $\mathrm{id} = (\mathrm{id}_X, \mathrm{id}_Z, \mathrm{id}_Y)$ is an endomorphism of Δ. The compositions of endomorphisms of triangles are defined in an obvious manner and is associative. We prove the following result.

Proposition 1.2.– *Let f be an endomorphism of a distinguished triangle $X \to Z \to Y \to X[1]$ in D with $f^n = \mathrm{id}$ for an integer $n > 0$. Then*

$$\mathrm{tr}(f_Z; Z) = \mathrm{tr}(f_X; X) + \mathrm{tr}(f_Y; Y).$$

§2. Proof

Let D and k be as above. We prove a more general result than 1.2. Let G be a group. A G-object in D is a pair (X, ρ) consisting of an object X of D and a k-algebra homomorphism $\rho : kG \to \mathrm{End}_{\mathfrak{A}}(X)$ where kG is the group algebra of G. We may denote $\rho(a)$ by a_X or simply a. Let Y be another G-object. An G-morphism or G-equivariant morphism from X to Y is a morphisms $f\colon X \to Y$ with $a_Y f = f a_X$ for all $a \in kG$. If X is an G-object define the central function

$$\chi_X\colon G \to \mathrm{End}_D(\mathbb{1}), \quad g \mapsto \mathrm{tr}(g; X).$$

We say that the distinguished triangle Δ is $G-$equivariant, if X, Y, and Z are equipped with actions $\rho_X\colon G \to \mathrm{Aut}_D(Z)$ (similarly for X and Y) and such that all morphisms (including the differential) are $G-$equivariant.

Theorem 2.1.– *If G is torsion and $X \to Z \to Y \to X[1]$ is $G-$equivariant, then as functions $G \to \mathrm{End}_D(\mathbb{1})$ we have*

$$\chi_Z = \chi_X + \chi_Y.$$

Proof. We may assume that G is finite. Let $\mathrm{Irr}kG$ be the set of isomorphism classes of irreducible $k-$representations of G. In D we have a natural G-equivariant isomorphism

$$X \simeq \coprod_{V \in \mathrm{Irr}kG} V \otimes_k S_V(X) \tag{2}$$

where $S_V(X) = \underline{\mathrm{Hom}}_{kG}(V, X)$ are certain objects and on which G acts trivially. To see this, consider the contravariant functor $D \to (k-\mathrm{mod})$ given by

$$\mathrm{Obj}(D) \ni Y \mapsto \mathrm{Hom}_{kG}\big(V, \mathrm{Hom}_D(Y, X)\big).$$

This is representable. Indeed if in the above definition we replace V by any finitely generated free kG-module M and consider the corresponding functor, we see immediately that the functor is representable by an object $S_M(X) =$ a finite direct sum of X. The general case follows from this and the fact that V is a finitely generated projective kG-module and hence the kernel (i.e. image) of a projector π on a free kG-module M. Since D is Karoubian, we can define $S_V(X) = \mathrm{coker}(\pi^*)$ where $\pi^*\colon S_M(X) \to S_M(X)$ is induced by π. This is easily seen to represent $S_V(X)$. Once we have these objects, the decomposition of X follows from the corresponding one for kG. It follows that the sequence

$$S_V(X) \to S_V(Z) \to S_V(Y) \to S_V(X[1])$$

being a direct summand of the original distinguished triangle is distinguished in D. Finally we note that by the above decomposition and k-linearity of trace we have

$$\mathrm{tr}(g, X) = \sum \chi_V(g)\mathrm{tr}(\mathrm{id}; S_V(X)) \tag{3}$$

where $\chi_V\colon G \to k$ is the usual character of V. Similarly for Z and Y. The result follows from this and 1.1. □

Proof of 1.2. Apply the result 2.1 with $G = \mathbb{Z}/n\mathbb{Z}$ and the action $m \mapsto f_Z^m$ (resp. $m \mapsto f_X^m, m \mapsto f_Y^m$) on Z (resp. X, Y). □

§3. Remark

We conclude this short note by indicating a corollary of the proof of 2.1. We let $\mathfrak{A}$ a Karoubian tensor category with $k \subseteq \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ where k is an algebraic closure of $\mathbb{Q}$. Define $\mathbb{Z}_{\mathfrak{A}}$ to be the subring (=subgroup) of $\mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ generated by all $\mathrm{tr}(\mathrm{id}; X)$ with X being dualizable in $\mathfrak{A}$.

Corollary 3.1.– *Let $f\colon X \to X$ be an endomorphism of a dualizable object in $\mathfrak{A}$ with $f^n = \mathrm{id}$ for an integer $n > 0$. Then $\mathrm{tr}(f; X) \in \mathrm{End}_{\mathfrak{A}}(\mathbb{1})$ is integral over $\mathbb{Z}_{\mathfrak{A}}$.*

Proof. Similar to the proof of 1.2 consider X with an action of $G = \mathbb{Z}/n\mathbb{Z}$. Note that in the category $\mathfrak{A}$ the decomposition (2) and the formula (3) hold with exactly the same proof. Since the element $\chi_V(g) \in k$ is integral over $\mathbb{Z}$, the result follows from (3). □

References

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